IMPROVEMENT OF GPS PHASE AMBIGUITY RESOLUTION USING PRIOR HEIGHT INFORMATION AS A QUASI-OBSERVATION

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GPS phase ambiguity resolution on L1 is always difficult for kinematic surveying, especially when the baseline is long. An attractive approach is to combine GPS observations with the information from other surveying systems or other sources to improve the ambiguity resolution. In bathymetric surveying, prior height information can be obtained from tide gauges. The present research is conducted to investigate how to use prior height information and how to obtain a robust solution. It addresses a method that uses prior height as a quasi-observation. This quasi-observation is then used in the adjustment along with GPS observations. In this contribution, an algorithm is first developed for the adjustment computation with the quasi-observation. The ability of the quasi-observation to improve the search technique is then studied in detail. The results show that not only can the quasi-observation strengthen the tests to eliminate incorrect solutions, but it can also advantageously change the structure of the ambiguity search space. The robustness of the method is discussed as well. Finally, field tests are conducted to demonstrate that the proposed approach is effective. The test results show that for bathymetric surveys in the St. Lawrence River, if tide interpolation technique is used and the squat, the heave and the draft of the ship are considered, an accuracy of the prior height (σ_{Hp}) of 10 or 20 cm can be selected.

La résolution des ambiguïtés sur les phases du GPS sur L1 est toujours difficile pour l'arpentage cinématique, particulièrement lorsque la ligne de base est longue. Une approche intéressante consiste à combiner les observations GPS à l'information d'autres systèmes d'arpentage ou d'autres sources pour améliorer la résolution des ambiguïtés. En levés bathymétriques, l'information antérieure sur la hauteur peut être obtenue à partir de mérigraphes. La présente recherche est entreprise pour étudier comment utiliser l'information antérieure sur la hauteur et comment obtenir une solution stable. Il s'agit d'une méthode qui utilise la hauteur antérieure comme une quasi-observation. Celle-ci est ensuite utilisée dans la compensation avec les observations GPS. Dans cette contribution, un algorithme est d'abord développé pour le calcul de compensation avec la quasi-observation. La capacité de la quasi-observation d'améliorer la technique de recherche est ensuite étudiée en détail. Les résultats montrent que non seulement la quasi-observation peut renforcer les tests pour éliminer les solutions incorrectes, mais elle peut aussi changer avantageusement la structure de l'espace de recherche des ambiguïtés. La stabilité de la méthode est également examinée. Enfin, des tests sur place sont entrepris pour démontrer que l'approche proposée est efficace. Les résultats des tests montrent que pour les levés bathymétriques dans le fleuve Saint-Laurent, si la technique d'interpolation de la marée est utilisée et si l'assise, le tanguage et le tirant d'eau du bateau sont considérés, une précision de la hauteur antérieure (σ_{Hp}) de 10 ou 20 centimètres peut être sélectionnée.

1. Introduction

The Laurentian Region of the Canadian Hydrographic Service (CHS) and the Canadian Coast Guard (CCG) conduct bathymetric surveys on the St. Lawrence River every year to check if its 300-km channel maintains its required nominal depth for navigation. The traditional method usually requires about 60-70 tide staffs to work together. It is costly and limited [Morneau et al. 1996]. In order to improve the bathymetric surveys, CHS implemented a network of 15 digital tide gauge stations along the St. Lawrence River in 1991 and CCG established a GPS-OTF network of three GPS reference stations in 1996. The acronym “OTF” means “On-the-fly”. This is an algorithm for GPS phase ambiguity resolution in kinematic mode. The network of tide gauges is called the Coastal and Oceanic Water Level Information System (COWLIS) and can provide regular tidal reading in real time [CHS 1997]. The GPS-OTF network can provide real-time GPS phase observations to bathymetric survey ships in the St. Lawrence River. These two systems can make the bathymetric surveying much more efficient by eliminating costly deployment (support and maintenance) of the tide staffs. In particular, the GPS technique can automatically and
accurately determine the water level when the bathymetric survey ship is working. However, when the GPS technique is used, the key problem is how to correctly resolve carrier phase ambiguities. During the bathymetric surveys, the distance between the GPS reference stations is more than 100 km and the distance between a reference station and a sounding ship can reach 75 km.

The aim of this research is to improve the methodology and algorithms of GPS-OTF ambiguity resolution over long distances to support, for example, bathymetric surveying. The improvement that will be studied in this paper is that may be obtained by using the height information from tide gauges to improve GPS phase ambiguity resolution.

Many studies have been done on how to use prior height information in GPS positioning [Lu et al. 1993; Weisenburger and Cannon 1997; Ueno et al. 2000]. Lu et al. [1993] solved the basic theoretical problem of inequality constraint, which can be used in height constraint. They showed that for single point positioning, the height information can clearly improve the solutions. The research of Lu et al. [1993] did not include the GPS phase ambiguity resolution.

Weisenburger and Cannon [1997] have also studied the use of prior height information. They came to the conclusion that the effect of constraints on ambiguity resolution is quite significant as the time to resolution was reduced between 2 to 10 times; only the height constraint is of much importance for decreasing the time to resolution, while the other constraints are more for quality assurance of the correct ambiguities; biased height constraints can hinder the ambiguity resolution process.

Ueno et al. [2000] considered two kinds of prior height information: one is from the tide gauge, the other is from former epochs. Their work focuses on two aspects: one is to transform the COWLIS height from chart datum to geodetic height and the other is to use height threshold values to validate the GPS height obtained from ambiguity fixed solutions.

In the present contribution, we discuss the method that employs the prior height as a quasi-observation and uses it in the adjustment along with GPS observations. With this method, the prior height can strengthen the residual tests, and at the same time it can still be used to strengthen the position tests. In this paper, we will first develop an algorithm for the adjustment computation with the quasi-observation. Then, we will discuss the improvement of the quasi-observation on the tests to eliminate incorrect solutions, the improvement of the quasi-observation on the structure of the ambiguity search space, and the robustness of the method.

2. Algorithm Using the Quasi-Observation

When the quasi-observation is used, one can carry out the adjustment for OTF directly using the following equation:

\[ L + \hat{\mathbf{V}} = \hat{\mathbf{A}} \hat{\mathbf{X}} \]  
(1a)

\[ L_H + \hat{\mathbf{V}}_H = A_H \hat{\mathbf{X}} \]  
(1b)

Where \( L \) denotes GPS between-receiver single-difference carrier phase observations corrected for potential integer ambiguity values, \( \hat{\mathbf{V}} \) denotes the vector of observations residuals, \( \mathbf{A} \) is the design matrix which essentially contains the receiver-satellite unit vectors, \( \hat{\mathbf{X}} \) is the vector of adjusted unknowns, \( L_H \) is the quasi-observation, \( \hat{\mathbf{V}}_H \) is the residual of \( L_H \), \( A_H \) is the design matrix for \( L_H \), and

\[ \hat{\mathbf{X}} = (\varphi \lambda h \Delta \delta T)^T \]  
(2)

\[ A_H = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \quad L_H = H_{\text{pr}} \]

where \( \varphi \), \( \lambda \) and \( h \) denote the latitude, longitude, and geodetic height of the unknown point, \( \Delta \delta T \) is the relative receiver clock error (parameter), and \( H_{\text{pr}} \) is the prior height. Let us note that the tide gauge data (in reference to the Chart Datum) have to be reduced to the GPS antenna height with respect to the ellipsoid. This reduction process is presented in Ueno et al. [2000].

Equation (1a) is actually the model without the quasi-observation and equation (1b) is the model that uses the quasi-observation. If the weight matrix of \( L \) is \( P \) and the weight matrix of \( L_H \) is \( P_H \), by least squares adjustment, one can obtain:

\[ \hat{\mathbf{X}}_H = (A^T P A_H + A_H^T P_H A_H)^{-1} (A^T P + A_H^T P_H L_H), \]

\[ \hat{Q}_{X_H} = (A^T P A_H + A_H^T P_H A_H)^{-1} \]

\[ \hat{\mathbf{V}}_H = A \hat{\mathbf{X}}_H - L,H \]  
(3)

\[ Q_{\hat{\mathbf{V}}_H} = Q_{\hat{\mathbf{V}}_H} = Q_{\hat{\mathbf{V}}_H} = Q_{\hat{\mathbf{V}}_H} A \]

where \( Q_{X_H} \) denotes the cofactor matrix of the estimate \( \hat{\mathbf{X}}_H \), \( Q_{\hat{\mathbf{V}}_H} \) denotes the cofactor matrix of the estimate \( \hat{\mathbf{V}}_H \), the subscript \( H \) indicates that the quasi-
observation $L_H$ has been used, and $Q_L = P^{-1}$ is the
cofactor matrix of $L$. In most cases, commercial
software is based on the model (1a). In order to eas-
ily upgrade the available software, we can use the
following algorithm. With the model (1a), one can obtain:

$$\begin{align*}
\hat{X} &= (A^T P A)^{-1} A^T P L, \\
\hat{V} &= A\hat{X} - L, \\
Q_X &= (A^T P A)^{-1} \\
Q_V &= Q_L - AQ_X A^T
\end{align*}$$

(4)

Qx, Qv, are respectively the cofactor matrices of the
estimates $\hat{X}$ and $\hat{V}$. Using sequential least squares the-
ory [Mikhail 1976] one can obtain:

$$\begin{align*}
\hat{X}_H &= \hat{X} + Q_{XH} A^T H P_{HPR} + A_H Q_X A_H^T (L_H - A_H \hat{X}_H), \\
Q_{XH} &= Q_X - Q_{XH} A^T H P_{HPR} + A_H Q_X A_H^T A_H Q_X
\end{align*}$$

(5)

where $Q_{HPR} = 1/P_{HPR}$, $P_{HPR}$ is the weight of the quasi
height observation $H_{PR}$. Let $q_{HH} = A_H^T Q_X A_H^T$ denote
the diagonal element of $Q_X$ corresponding to $H$, $q_H$
$= Q_H A_H^T$ denote the column of $Q_X$ corresponding to
$H$. $H$ denotes the estimate of height without regard to prior height information. Considering equation
(2), equation (5) can be rewritten as:

$$\begin{align*}
\hat{X}_H &= \hat{X} + q_{HPR} H_{PR} - H \\
Q_{XH} &= Q_X - q_{HPR} Q_{HPR} + q_{HH}
\end{align*}$$

(6)

Let

$$\begin{align*}
\Delta X &= q_{HPR} H_{PR} - H \\
\Delta Q &= - q_{HPR} Q_{HPR} + q_{HH}
\end{align*}$$

(7)

From equation (6), and considering equations
(3) and (4), we have:

$$\begin{align*}
\hat{X}_H &= \hat{X} + \Delta X, \\
Q_{XH} &= Q_X + \Delta Q \\
\hat{V}_H &= \hat{V} + A\Delta X, \\
Q_{VH} &= Q_V - A\Delta Q A^T
\end{align*}$$

(8)

Equation (8) actually denotes the changes caused by the prior height.

In our project, the least squares ambiguity search technique suggested by Hatch [1990] is
used. In this OTF technique, we must compute the
position $\hat{X}_H$ and $\hat{V}_H$ for every potential solution in
search space. Using equations (7) and (8), we can
easily get $\hat{X}_H$ and $\hat{V}_H$ and perform the tests for the
search of ambiguities.

3. Improvement of Ambiguity Search Tests Using the Quasi-Observation

In the process of integer ambiguity search, having incorrect solutions means that the solutions contain deviations. These deviations are one or more times the wavelength of the carrier phase and are usually much larger than the other errors that affect the observations; thus they will behave as outliers. One kind of ambiguity search test eliminates the incorrect solutions by detecting this kind of outlier in the observations residuals. If these tests have a stronger ability to detect outliers, we can ensure that their ability to eliminate incorrect solutions will also be stronger. From the theory of data snooping [Baarda 1968], one can know that the ability of the tests to detect outliers will depend on the redundancy number. If one observation has a larger redundancy number, the outlier will be more easily detected. In our case, it can easily be shown and understood that the redundancy number of an observation will be increased if the quasi-observation (the prior height) is added to the adjustment (see Appendix A). This means that the quasi-observation can increase the ability of residual tests to eliminate incorrect solutions.

In order to understand how much the quasi-observation will increase the redundancy number, we provide an example in which seven satellites are in view. The results are presented in Tables 1, 2 and 3. In our algorithm, the L1 ambiguity search starts from five primary satellites and then the satellites from the second group are added to the ambiguity search one by one. Hence we compute the redundancy number separately for five, six and seven primary satellites.

Data from these tables come from the field test DI4 which will be described in section 6. The length of the baseline is about 45 km. "Without $H_{PR}$" means that the quasi-observation is not used. "With $\sigma_{H_{PR}} = 10$ cm" means that the quasi-observation has been used and its associated variance is (10 cm)$^2$. From the previous tables, we can see that for some satellites, the redundancy number can be greatly improved. For example, for satellite #9, if no prior height is used, its redundancy number is only 0.003
(Table 1). If prior height "With $\sigma_{H_{pr}} = 10$ cm" is used, its redundancy number becomes 0.121. The ability of the tests to detect outliers will be improved significantly. But for some satellites the improvement are very small so that we can not see the increment in three digits. From Tables 1 to 3, we can also note that the fewer satellites that are in view, the more obvious the improvement will be. In this example, when five satellites are used, the improvement is significant. When seven satellites are used, only the improvement on satellite #5 is significant.

4. Improvement of the Structure of the Ambiguity Search Space

If the quasi-observation is not used, from equation (4), residuals can be expressed as:

$$\hat{\mathbf{V}} = - Q_v \mathbf{P} \mathbf{L}$$

The vector $\mathbf{L}$ will consist of the phase observations and a potential ambiguity solution, that is,

$$\mathbf{L} = \mathbf{L}_\phi + \mathbf{X}_N$$

where $\mathbf{L}_\phi$ denotes the ambiguous GPS phase observations and $\mathbf{X}_N$ denotes the potential ambiguity solution. Therefore, the observation residuals can be expressed as:

$$\hat{\mathbf{V}} = - Q_v \mathbf{P}(\mathbf{L}_\phi + \mathbf{X}_N)$$

For the residual tests, the correct ambiguity solution $\mathbf{X}_N$ should pass them theoretically. But the incorrect solutions $\mathbf{X}_N + d\mathbf{X}_N$ can also pass the same residual tests if

$$Q_v \mathbf{P} d\mathbf{X}_N = 0$$

by:

$$\hat{\mathbf{V}}_d = - Q_v \mathbf{P}(\mathbf{L}_\phi + \mathbf{X}_N + d\mathbf{X}_N) = - Q_v \mathbf{P}(\mathbf{L}_\phi + \mathbf{X}_N) - Q_v \mathbf{P} d\mathbf{X}_N = - Q_v \mathbf{P}(\mathbf{L}_\phi + \mathbf{X}_N) = \hat{\mathbf{V}}$$

where $\hat{\mathbf{V}}_d$ denotes the residuals corresponding to the potential solution $\mathbf{X}_N + d\mathbf{X}_N$. In this situation, it is impossible for residual tests to eliminate the incorrect solutions. Whether this kind of incorrect solution can be eliminated or not will depend on the position tests. But if the quasi-observation is used, the situation will be changed, because for the same incorrect solution one will have:

$$Q_v H \mathbf{P} d\mathbf{X}_N \neq 0$$

This means that the detection of the incorrect solutions can be improved, although some other incorrect solutions will perhaps satisfy:

$$Q_v H \mathbf{P} d\mathbf{X}_N = 0$$

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Table 1: Redundancy number with and without the quasi-observation (5 primary satellites).

<table>
<thead>
<tr>
<th></th>
<th>Sat #23</th>
<th>Sat #8</th>
<th>Sat #21</th>
<th>Sat #9</th>
<th>Sat #1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without $H_{pr}$</td>
<td>0.563</td>
<td>0.380</td>
<td>0.042</td>
<td>0.003</td>
<td>0.015</td>
</tr>
<tr>
<td>With $\sigma_{H_{pr}} = 30$ cm</td>
<td>0.576</td>
<td>0.380</td>
<td>0.145</td>
<td>0.037</td>
<td>0.076</td>
</tr>
<tr>
<td>With $\sigma_{H_{pr}} = 10$ cm</td>
<td>0.607</td>
<td>0.382</td>
<td>0.397</td>
<td>0.121</td>
<td>0.226</td>
</tr>
</tbody>
</table>

Table 2: Redundancy number with and without the quasi-observation (6 primary satellites).

<table>
<thead>
<tr>
<th></th>
<th>Sat #23</th>
<th>Sat #8</th>
<th>Sat #21</th>
<th>Sat #9</th>
<th>Sat #1</th>
<th>Sat #29</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without $H_{pr}$</td>
<td>0.596</td>
<td>0.381</td>
<td>0.174</td>
<td>0.029</td>
<td>0.629</td>
<td>0.193</td>
</tr>
<tr>
<td>With $\sigma_{H_{pr}} = 30$ cm</td>
<td>0.598</td>
<td>0.381</td>
<td>0.215</td>
<td>0.047</td>
<td>0.630</td>
<td>0.222</td>
</tr>
<tr>
<td>With $\sigma_{H_{pr}} = 10$ cm</td>
<td>0.610</td>
<td>0.382</td>
<td>0.399</td>
<td>0.129</td>
<td>0.637</td>
<td>0.351</td>
</tr>
</tbody>
</table>

Table 3: Redundancy number with and without the quasi-observation (7 primary satellites).

<table>
<thead>
<tr>
<th></th>
<th>Sat #23</th>
<th>Sat #8</th>
<th>Sat #21</th>
<th>Sat #9</th>
<th>Sat #1</th>
<th>Sat #29</th>
<th>Sat #5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without $H_{pr}$</td>
<td>0.608</td>
<td>0.462</td>
<td>0.597</td>
<td>0.206</td>
<td>0.657</td>
<td>0.332</td>
<td>0.138</td>
</tr>
<tr>
<td>With $\sigma_{H_{pr}} = 30$ cm</td>
<td>0.615</td>
<td>0.469</td>
<td>0.599</td>
<td>0.207</td>
<td>0.657</td>
<td>0.338</td>
<td>0.148</td>
</tr>
<tr>
<td>With $\sigma_{H_{pr}} = 10$ cm</td>
<td>0.660</td>
<td>0.521</td>
<td>0.611</td>
<td>0.213</td>
<td>0.657</td>
<td>0.381</td>
<td>0.210</td>
</tr>
</tbody>
</table>
In order to investigate how much the quasi-observation will improve the situation, we employ an example. We can find all the incorrect solutions that satisfy equation (12). Then we can find all the incorrect solutions that satisfy equation (14) to investigate the change introduced by the use of the quasi-observation. Data used in this example also come from the field test D14 (see section 6). The moving receiver in this case is 45 km from the reference station. The results are shown in Figure 1.

Due to the errors in the observations, it will be difficult to detect the incorrect solutions with residual tests if

\[ Q_Y P dX_N \leq \delta_Y \]  
(15)

\[ dX_N^T P Q_Y P dX_N \leq \delta_{T2} \]  
(16)

where \( Q_Y P dX_N \) is the deviation contained in \( \hat{V} \), which is caused by the deviation in the incorrect ambiguity solution, \( \delta_Y \) denotes the minimum detectable outlier in \( \hat{V} \) in using residual tests, \( dX_N^T P Q_Y P dX_N \) is the deviation contained in \( \hat{V}^T PV \), and \( \delta_{T2} \) denotes the minimum detectable deviation contained in \( \hat{V}^T PV \) using residual tests. To derive equation (16), we used the fact that \( (PQ_\lambda) (PQ_\lambda) = (PQ_\lambda) \), as demonstrated in Appendix C. Hence in this example, we actually test potential solutions by the inequality of equations (15) and (16).

Figure 1(a) shows the potential solutions remaining in the search space if no prior height information is used. The points, except the one on the origin, denote the incorrect solutions. The point on the origin is the correct solution. Figure 1(b) shows the result when prior height is used. From Figure 1, we can see that if we do not use the quasi-observation, there are 22 incorrect solutions that theoretically cannot be eliminated by the residual tests. But if prior height is used as a quasi-observation with \( \sigma_{\text{Hpr}} = 10 \text{ cm} \), there are only three incorrect solutions that cannot be eliminated by the residual tests. This result shows that the quasi-observation can improve the detection of incorrect solutions significantly and hence improve the structure of integer ambiguity search.

5. Robustness of Ambiguity Resolution Using the Quasi-Observation

It is not difficult to understand that the use of the quasi-observation can improve the ambiguity resolution since the quasi-observation can increase the information provided for ambiguity resolution. The key problem will be how to handle the bias in the quasi-observation, that is, the robustness of ambiguity resolution using the quasi-observation, because in many situations one cannot ensure that one can obtain a reliable prior height. For bathymetric surveys in the St. Lawrence River, although

Figure 1(a): Potential solutions left in the search space without prior height.

Figure 1(b): Potential solutions left in the search space with prior height (\( \sigma_{\text{Hpr}} = 10 \text{ cm} \)).
the COWLIS tide gauges are accurate and reliable, the prior height of the working ship interpolated from tide gauges can become less accurate due to, for example, the distance of the working ship from the tide gauge stations, the height reduction from Chart Datum to the ellipsoid, the water wave state, and the movement of the ship. For inaccurate prior information, Zhu and Wang [2000] defined an admissible criterion for the Bayesian estimation. The basic idea of this criterion is that inaccurate prior information can be used only if it can make the Bayesian estimation have a better accuracy than can be obtained without using any prior information. The accuracy can be measured by Mean Squared Error (MSE) or variance [Zhu 1991; Wang 1987; Montgomery and Peck 1992]. In this case, if the quasi-observation with bias can improve the ambiguity resolution, we will consider it as being useful and the corresponding method as being robust. A relationship between the accuracy of the position parameters and the biases of the prior height information to investigate indirectly the effect of the biases on the ambiguity resolution is also established.

There are two kinds of biases that prior height information might contain: one is the bias of the value of prior height \( H_{pr} \) (the bias in the quasi-observation), the other is the bias of the accuracy of the prior height (the bias in the weight of the quasi-observation). If there is a bias \( \varepsilon H_{pr} \) in the prior height, its MSE will be:

\[
\text{MSE}(H_{pr}) = \sigma^2 Q_{H_{pr}} + \varepsilon H_{pr}^2
\]  

where \( Q_{H_{pr}} \) is the true value of the cofactor of the prior height and \( \sigma^2 \) is the variance factor.

Rearranging the terms in equation (6), one can obtain:

\[
\hat{X}_H = \hat{X} + q_H \frac{H_{pr} - H}{Q_{H_{pr}} + q_{HH}} + q_H \frac{H_{pr} - H}{Q_{H_{pr}} + q_{HH}}
\]

\[
= \left( \hat{X} - q_H \frac{\hat{A}_H}{Q_{H_{pr}} + q_{HH}} \right) + q_H \frac{\hat{A}_H}{Q_{H_{pr}} + q_{HH}} + q_H \frac{H_{pr} - H}{Q_{H_{pr}} + q_{HH}}
\]

\[
= \left( \hat{X} - q_H \frac{\hat{A}_H}{Q_{H_{pr}} + q_{HH}} \right) + q_H \frac{\hat{A}_H}{Q_{H_{pr}} + q_{HH}} + q_H \frac{H_{pr} - H}{Q_{H_{pr}} + q_{HH}}
\]

\[
(18)
\]

Assuming that \( \hat{X} \) and \( H_{pr} \) are not correlated, the MSE of \( \hat{X}_H \) will be:

\[
\text{MSE}(\hat{X}_H) = \sigma^2 \left( 1 - q_H \frac{\hat{A}_H}{Q_{H_{pr}} + q_{HH}} \right)
\]

\[
Q_X \left( 1 - q_H \frac{\hat{A}_H}{Q_{H_{pr}} + q_{HH}} \right) + \frac{\sigma^2 Q_{H_{pr}} + \varepsilon H_{pr}^2}{Q_{H_{pr}} + q_{HH}} q_H
\]

\[
(19)
\]

where \( Q_{H_{pr}} \) is the value adopted for the cofactor of the prior height.

If there is a bias in the accuracy of the prior height, then \( Q_{H_{pr}} \neq Q_{H_{pr}}^* \). Let

\[
\sigma^2 Q_{H_{pr}} = \sigma^2 Q_{H_{pr}}^* + \varepsilon Q_{H_{pr}}
\]

and one will have (see Appendix B):

\[
\text{MSE}(\hat{X}_H) = \sigma^2 Q_X - q_H \frac{\sigma^2}{Q_{H_{pr}} + q_{HH}}
\]

\[
+ \varepsilon H_{pr}^2 \frac{\varepsilon H_{pr} - \varepsilon Q_{H_{pr}}}{(Q_{H_{pr}} + q_{HH})^2}
\]

where \( \varepsilon Q_{H_{pr}} \) denotes the bias in the accuracy of the prior height. In the above equation, the first term denotes the accuracy without using prior height information, the second term denotes the improvement of the prior height on the accuracy of the estimation, and the third term is the loss caused by the bias in the prior height. The sum of the second and third terms is the net effect of the prior height on the estimation. If

\[
q_H \frac{\sigma^2}{Q_{H_{pr}} + q_{HH}} - \frac{\varepsilon H_{pr} - \varepsilon Q_{H_{pr}}}{(Q_{H_{pr}} + q_{HH})^2} > 0
\]

then the prior height will cause the estimate to have a higher accuracy and hence play a useful role in the adjustment. From equation (22) one will have:

\[
\varepsilon H_{pr}^2 - \varepsilon Q_{H_{pr}} < \sigma^2 (Q_{H_{pr}} + q_{HH})
\]

or

\[
\varepsilon H_{pr}^2 - \varepsilon Q_{H_{pr}} < \sigma^2 H_{pr} + \sigma^2 H
\]

Here, \( \sigma^2 H_{pr} = \sigma^2 q_{HH} \) is the variance adopted for the prior height and \( \sigma^2 H = \sigma^2 q_{HH} \) is the variance of the
estimate of height without using prior information. From equation (23), three cases can be considered separately:

1) The case $\epsilon Q_{\text{hp}} = 0$ and $\epsilon H_{\text{pr}} \neq 0$. In this situation, we will have:

$$\epsilon H_{\text{pr}}^2 < \sigma_{H_{\text{pr}}}^2 + \sigma_H^2$$

(24)

This means that if the weight is realistic, the bias allowed in the value of the prior height will be limited by equation (24). For example, if the prior height has a mean error $\sigma_{H_{\text{pr}}} = 5$ cm and the accuracy of GPS positioning is $\sigma_H = 3$ cm, then the maximum bias allowed will be $\epsilon H_{\text{pr}} < \sqrt{5^2 + 3^2} = 6$ cm. In other words, if the bias is less than 6 cm, the prior height will still be useful. But if the bias is larger than 6 cm, the prior height will play a negative role in the ambiguity resolution and will be considered useless.

2) The case $\epsilon H_{\text{pr}} = 0$ and $\epsilon Q_{\text{hp}} \neq 0$. In this situation, considering also equation (20), we will have:

$$-\epsilon Q_{\text{hp}} < \sigma^2(Q_{\text{hp}} + q_{\text{hp}}) = \sigma^2(Q_{\text{hp}} + q_{\text{hp}}) + \epsilon Q_{\text{hp}}$$

$$\epsilon Q_{\text{hp}} > -\frac{\sigma_{Q_{\text{hp}}}^2 + \sigma_H^2}{2}$$

(25)

where $\sigma_{Q_{\text{hp}}}^2 = \sigma^2 Q_{\text{hp}}$. From equation (25), two conclusions can be drawn: the first is that if the adopted accuracy is better than the true accuracy, that is, $\epsilon Q_{\text{hp}} < 0$, then the deviation will be limited by equation (25). For example, if the prior height has a mean error $\sigma_{H_{\text{pr}}} = 5$ cm and the accuracy of GPS positioning is $\sigma_H = 3$ cm, the maximum bias allowed in the variance will be $\epsilon Q_{\text{hp}} = -(25 + 9)/2 = -17$ cm, and $Q_{H_{\text{pr}}} = \sqrt{25^2 + 17^2} = \sqrt{625 + 289} = 3 cm$. This means that if the true accuracy of the prior height is 5 cm and one uses 3 cm as the accuracy, the prior height will not play a negative role in the adjustment. But if one uses a variance that is less than (3 cm)$^2$, it will play a negative role in the adjustment. The second conclusion that can be drawn is that the case in which the adopted accuracy is worse than the true accuracy ($\epsilon Q_{\text{hp}} > 0$) is always allowed. This situation occurs when the observation having a higher accuracy is used as a less accurate observation. For the previous situation, that is, the prior height has a mean error $\sigma_{H_{\text{pr}}} = 5$ cm, we can adopt $\sigma_{H_{\text{pr}}} = 6, 7, 10$ cm or any value larger than 5 cm as its accuracy, and the result will usually be improved by the prior height.

3) The case $\epsilon H_{\text{pr}} \neq 0$ and $\epsilon Q_{\text{hp}} \neq 0$. Considering also equation (20), in this situation, we will have:

$$\epsilon H_{\text{pr}}^2 - \epsilon Q_{\text{hp}} < \sigma_{H_{\text{pr}}}^2 + \sigma_H^2$$

(26a)

and considering equation (20), it can be written as

$$\epsilon H_{\text{pr}}^2 - 2\epsilon Q_{\text{hp}} < \sigma_{H_{\text{pr}}}^2 + \sigma_H^2$$

(26b)

From these two equations, we can see that if $\epsilon H_{\text{pr}} \neq 0$, we can make $\epsilon Q_{\text{hp}} > 0$ to compensate for the influence of $\epsilon H_{\text{pr}}$. If $\epsilon H_{\text{pr}}$ has a large value, we can give $\epsilon Q_{\text{hp}}$ a large positive value to satisfy equation (26). If $\epsilon Q_{\text{hp}} > \epsilon H_{\text{pr}}^2 / 2$, then the above equation will always be satisfied. From equation (21), we can see that if $\epsilon Q_{\text{hp}} = \epsilon H_{\text{pr}}^2$, the third term, that is, the loss caused by the biases, will be 0.

Based on these three situations, we can adopt the following robust method to determine the weight of the prior height:

1) On the basis of the prior information, determine the variance of the prior height $\sigma_{H_{\text{pr}}}^2$.

2) Based on the working conditions and surroundings, determine the bias that might be contained in the prior height.

3) According to the working conditions, if there is a large possibility that the bias exists, we can use:

$$\sigma_{H_{\text{pr}}}^2 = \sigma_{H_{\text{pr}}}^2 + \epsilon H_{\text{pr}}^2$$

(27)

as the variance of the quasi-observation. However, if there is a small possibility that the bias exists, we can use:

$$\sigma_{H_{\text{pr}}}^2 \geq \sigma_{H_{\text{pr}}}^2 + \epsilon H_{\text{pr}}^2 / 2$$

(28)

6. Analysis of Field Test Results

In order to investigate the reliability of using GPS for bathymetric surveys in the St. Lawrence River, field tests were conducted by CHS and CCG. Four tests, 1015, 1017, D11 and D14, are employed in this research.

Tests 1015 and 1017 were conducted on October 15 and October 17, 1999 in the region of Quebec City. The distance of the working ship to the COWLIS station in the port of Quebec was about 5 km. The distance to the GPS reference station at
Launon was about 7 km. For these tests, the working ship was only 9 m in length. Tests D11 and D14 were conducted in October, 1998 at Lac St. Pierre. The distance of the working ship in test D11 to the GPS reference station at Trois-Rivières was about 45 km. The distance to the COWLIS gauge station in the lake was about 8 km. The distance of the working ship in test D14 to the GPS reference station at Trois-Rivières was about 35 km. The distance to the COWLIS gauge station in the lake was about 5 km. The working ships for tests D11 and D14 was 35 m long. We mention that solar activity, and consequently the ionospheric effect on GPS observations, during the year 1998 was smaller than during the year 1999.

All datasets contain dual frequency phase and P(Y) code GPS observations, and the observation rate is one second. In the first step of our algorithm we use dual frequency data to form the widelane (L4) combinations. The L4 combination provided an 86-cm wavelength (in comparison the L1 signal has a 19-cm wavelength). The ambiguity resolution on L4 observations is then facilitated. Then, in the second step of our algorithm, we use the L4 positioning results to help fix the L1 phase ambiguities. If at the first epoch, the L1 phase ambiguities are not fixed, we will keep all potential L1 ambiguity combinations and use the phase observations of the second epoch to try to identify the correct set of L1 phase ambiguities. We use five epochs to try to correctly solve the L1 phase ambiguities, so the initialization time is 5 seconds (a relatively short period of time for bathymetric surveys). In order to study the success rate of the algorithm, we performed initialization every 5 seconds. The success rate, reported in table 4, is the ratio of the number of times that the ambiguity initializations are successful compared to the number of all initializations.

As mentioned above, the water level on the St. Lawrence River is accessible from the COWLIS network. This network provides regular readings at a 3-minute interval. However, in these tests, the COWLIS readings from the tide gauge stations are at a 15-minute interval. The height of water level at any second is interpolated from the COWLIS readings. Because the water level at different places of the river will be different, a spatial interpolation using data from two tide gauge stations is needed. From the height of the water level and the squat, the heave and the draft of the ship, one can derive the height \( H_p \) of the working ship [Ueno et al. 2000]. This height can be used as the prior height for GPS positioning.

The key problem is the determination of the accuracy of the prior height. Ueno et al. [2000] showed that the difference of the height between the tide gauge and GPS are \( A \pm B \), where \( A = -6.8 \) to 10.2 cm is the mean of the difference (bias) and \( B = 1.5 \) to 3.0 cm is the RMSE. The value of \( A \) and \( B \) will depend upon whether the interpolation among the tide gauge stations is used or not, on the accuracy of the determination of the squat, the heave and the draft of the working ship, and the distance of the working ship from the tide gauge station. Thus we can consider that the accuracy of the height from the tide gauge is about 3 cm, that is, \( B = 3 \) cm. Now the difficulty is to determine the bias \( A \). If the bias is less than 5 cm, according to equation (27) we should take \( \sigma_{H_p} \geq \sqrt{3^2 + 5^2} = 6 \) cm as the accuracy of the quasi-observation. If the bias is about 5-10 cm, we should take \( \sigma_{H_p} = \sqrt{3^2 + 10^2} = 10 \) cm as the accuracy of the quasi-observation. If the bias is about 10 to 20 cm, one should take \( \sigma_{H_p} = \sqrt{3^2 + 20^2} = 20 \) cm as the accuracy. For bathymetric surveys in the St. Lawrence River, we believe there is small possibility for the bias to reach more than 30 cm. Even if the bias reaches 30 cm, according to equation (28), we can still take \( \sigma_{H_p} = \sqrt{3^2 + 30^2 + 2} = 20 \) cm as the accuracy. But in this research we still consider the cases that the biases are larger than 20 cm. So we will investigate \( \sigma_{H_p} = 3, 6, 10, 20, 30 \) and 40 cm for each test. The value of 3 cm corresponds to the case where the bias is zero. The values 30 and 40 cm correspond to the cases where the biases are larger than 20 cm. The success rate of GPS L1 ambiguity resolution using the quasi-observation are listed in Table 4.

The last column shows the results without using prior height information. The results in the “Height Val.” column are obtained from the method described in [Ueno et al. 2000], where prior height is only used to validate the GPS ambiguity fixed solutions. The results of the method presented in this paper show that if reasonable weight or accuracy is selected, the quasi-observation can improve the ambiguity resolution significantly. But if the weight is not accurate or the prior height contains a bias, the results will be adversely influenced. This is shown clearly by test 1017. In that test, there is a bias of about -10 cm in the prior height. If we take 3 cm as the accuracy, the results become clearly worse: the success rate drops to 73%. However, if we remove this bias, the success rate returns to 98%. From equation (27), we know that for a bias of about 10 cm we should use \( \sigma_{H_p} = \sqrt{3^2 + 10^2} = \).
Table 4: Success rate of GPS L1 ambiguity resolution using the quasi-observation for a 5 second initialization period.

<table>
<thead>
<tr>
<th>Tests</th>
<th>$c_{h_{pr}}$</th>
<th>3 cm</th>
<th>6 cm</th>
<th>10 cm</th>
<th>20 cm</th>
<th>30 cm</th>
<th>40 cm</th>
<th>Height Val.</th>
<th>No Height</th>
</tr>
</thead>
<tbody>
<tr>
<td>1015</td>
<td>73%</td>
<td>82%</td>
<td>92%</td>
<td>95%</td>
<td>91%</td>
<td>90%</td>
<td>89%</td>
<td>69%</td>
<td></td>
</tr>
<tr>
<td>1017</td>
<td>73%</td>
<td>90%</td>
<td>95%</td>
<td>96%</td>
<td>93%</td>
<td>92%</td>
<td></td>
<td>92%</td>
<td>61%</td>
</tr>
<tr>
<td>1017(-10cm)</td>
<td>98%</td>
<td>98%</td>
<td>99%</td>
<td>98%</td>
<td>98%</td>
<td>97%</td>
<td></td>
<td>96%</td>
<td>61%</td>
</tr>
<tr>
<td>D14</td>
<td>85%</td>
<td>90%</td>
<td>91%</td>
<td>92%</td>
<td>91%</td>
<td>90%</td>
<td></td>
<td>89%</td>
<td>63%</td>
</tr>
<tr>
<td>D11</td>
<td>81%</td>
<td>84%</td>
<td>84%</td>
<td>77%</td>
<td>72%</td>
<td>66%</td>
<td></td>
<td>64%</td>
<td>50%</td>
</tr>
</tbody>
</table>

10 cm. In fact, in all four tests where $c_{h_{pr}} = 6$, 10 or 20 cm are used, the ambiguity resolution success rate improved. But in the situation where $c_{h_{pr}} = 6$ cm, the results can easily be influenced by biases (for instance in the case 1017). Therefore for robustness, $c_{h_{pr}} = 10$ or 20 cm are more appropriate for bathymetric surveys in the St. Lawrence River, if tide interpolation technique is used and the squat, the heave and the draft are taken into account. When $c_{h_{br}} = 30$ or 40 cm are used, results are improved too, but the results are not as good as in the cases where $c_{h_{pr}} = 10$ or 20 cm are used.

7. Conclusions

The prior height can be taken as a quasi-observation and used along with GPS observations. It can strengthen the residual tests, especially when there are fewer satellites in the sky. It can also advantageously change the structure of the ambiguity search space: this will theoretically leave fewer incorrect solutions in the search space. The effect of these two aspects is to improve the ambiguity resolution significantly.

By using the algorithm developed in this paper, the computation is very simple and begins with the results without a prior height. Only a small change needs to be made to standard software to compute the GPS solution with the use of a prior height.

If the prior height is used as a quasi-observation, its weight must be determined properly. Usually, if the adopted accuracy is better than it really is, it may introduce a negative role in the computation. However, if the adopted accuracy is poorer than it really is, the prior height will improve the ambiguity resolution in most cases, at least it will not make the ambiguity resolution worse. For bathymetric surveys in the St. Lawrence River, if tide interpolation technique is used and the squat, the heave and the draft are considered, an accuracy of the prior height ($c_{h_{pr}}$) of 10 or 20 cm can be selected.

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References


Appendix A:
Proof of the Improvement of the Redundancy Number Using the Quasi-Observation

For the observation equation without using prior height (model 1a), the element of redundancy number of an observation can be expressed as

\[ r_i = (Q_vP)_{ii} \]  \hspace{1cm} (A1)

where

\[ Q_v = Q_L - A(A^T P A)^{-1} A^T \]  \hspace{1cm} (A2)

\( Q_v \) is the cofactor matrix of residuals \( \hat{V} \), and \( P \) is the weight matrix of observations \( L \), see equation (4). If \( n \) denotes the number of observations and \( u \) denotes the number of unknowns, we will have:

\[ 0 \leq r_i \leq 1 \]

\[ r = \sum r_i = n - u \]  \hspace{1cm} (A3)

If a quasi-observation is used, the corresponding mathematical model will become the summation of models (1a) and (1b). The observation number will be \( n+1 \), and the redundancy number will become:

\[ r_H = \sum r_{Hi} = n + 1 - u > r \]  \hspace{1cm} (A4)

The cofactor matrix \( Q_{vH} \) will become:

\[ Q_{vH} = Q_L - A(A^T P A + A_{H}^T P A_{H})^{-1} A^T \geq Q_L - A(A^T P A)^{-1} A^T = Q_v \]  \hspace{1cm} (A5)

and the element of the redundancy number will become:

\[ r_{Hi} = (Q_{vH} P)_{ii} \geq (Q_v P)_{ii} = r_i \]  \hspace{1cm} (A6)

where the subscript \( H \) is related to the quasi-observation \( L_{Hi} \).

Appendix B:
Proof of Equation (21)

In recalling the definition in the paper (below equation (5) and equation (2)), we have:

\[ q_{HiH} = A_{H}^T Q_{HiH} A_{H}^T \]

\[ X = (\varphi \lambda \Delta \Lambda \eta) \]

\[ A_{H} = (0 \ 0 \ 1 \ 0) \]

\[ L_{Hi} = H_{pr} \]

Equation (19) can be developed as follow:

\[ \text{MSE}(X) = \sigma^2 \left( I - q_{Hi} \frac{A_{H}^T}{Q_{HiH} + q_{HiH}} \right) Q_X \]

\[ \left( I - q_{Hi} \frac{A_{H}^T}{Q_{HiH} + q_{HiH}} \right) \frac{\sigma^2 Q_{HiH} + \varepsilon H_{pr}^2}{Q_{HiH} + q_{HiH}} \]

\[ = \sigma^2 \left( \frac{Q_X - q_{Hi} A_{H}^T Q_{HiH} A_{H}^T}{Q_{HiH} + q_{HiH}} \right) \frac{Q_X}{Q_{HiH} + q_{HiH}} \]

\[ + q_{Hi} H_{pr} + \varepsilon H_{pr}^2 \]

\[ + \frac{\sigma^2 Q_{HiH} + \varepsilon H_{pr}^2}{Q_{HiH} + q_{HiH}} \]  \hspace{1cm} (B1)

Considering equation (20):

\[ \sigma^2 Q_{HiH} = \sigma^2 Q_{HiH} + \varepsilon Q_{HiH} \]  \hspace{1cm} (B3)

and introducing equations (B1) and (B3) into equation (B2) we have:
MSE(\(\hat{X}_H\))

\[
= \sigma^2 \left( Q_X - q_Hq_H^T \right) + q_H \frac{T}{Q_{Hpr} + q_{HH}} \left( \frac{q_{HH}}{Q_{Hpr} + q_{HH}} q_H^T \right) + q_H \frac{T}{Q_{Hpr} + q_{HH}} \left( \frac{q_{HH}}{Q_{Hpr} + q_{HH}} q_H^T \right) + q_H \frac{T}{Q_{Hpr} + q_{HH}} \left( \frac{q_{HH}}{Q_{Hpr} + q_{HH}} q_H^T \right) + q_H \frac{T}{Q_{Hpr} + q_{HH}} \left( \frac{q_{HH}}{Q_{Hpr} + q_{HH}} q_H^T \right) + q_H \frac{T}{Q_{Hpr} + q_{HH}} \left( \frac{q_{HH}}{Q_{Hpr} + q_{HH}} q_H^T \right)
\]

\[
= \sigma^2 \left( Q_X - 2q_Hq_H^T \right) + q_H \frac{T}{Q_{Hpr} + q_{HH}} \left( \frac{q_{HH}}{Q_{Hpr} + q_{HH}} q_H^T \right) + q_H \frac{T}{Q_{Hpr} + q_{HH}} \left( \frac{q_{HH}}{Q_{Hpr} + q_{HH}} q_H^T \right) + q_H \frac{T}{Q_{Hpr} + q_{HH}} \left( \frac{q_{HH}}{Q_{Hpr} + q_{HH}} q_H^T \right) + q_H \frac{T}{Q_{Hpr} + q_{HH}} \left( \frac{q_{HH}}{Q_{Hpr} + q_{HH}} q_H^T \right) + q_H \frac{T}{Q_{Hpr} + q_{HH}} \left( \frac{q_{HH}}{Q_{Hpr} + q_{HH}} q_H^T \right)
\]

\[
= \sigma^2 Q_X - \frac{\sigma^2 q_Hq_H^T}{Q_{Hpr} + q_{HH}} + q_H \frac{T}{Q_{Hpr} + q_{HH}} \left( \frac{q_{HH}}{Q_{Hpr} + q_{HH}} q_H^T \right) + q_H \frac{T}{Q_{Hpr} + q_{HH}} \left( \frac{q_{HH}}{Q_{Hpr} + q_{HH}} q_H^T \right) + q_H \frac{T}{Q_{Hpr} + q_{HH}} \left( \frac{q_{HH}}{Q_{Hpr} + q_{HH}} q_H^T \right) + q_H \frac{T}{Q_{Hpr} + q_{HH}} \left( \frac{q_{HH}}{Q_{Hpr} + q_{HH}} q_H^T \right) + q_H \frac{T}{Q_{Hpr} + q_{HH}} \left( \frac{q_{HH}}{Q_{Hpr} + q_{HH}} q_H^T \right)
\]

This is the proof of equation (21):

\[
\text{MSE} (\hat{X}_H) = \sigma^2 Q_X - q_Hq_H^T \frac{\sigma^2}{Q_{Hpr} + q_{HH}} + \frac{\epsilon H_{pr}^2 - \epsilon q_{pr}}{Q_{Hpr} + q_{HH}}
\]

\[
\text{(B4)}
\]

**Appendix C : Proof of**

\((PQ_v)(PQ_v) = (PQ_v)\)

From equation (4), we know that

\[Q_v = Q_L - AQ_XA^T\]

then

\[PQ_v = PQ_L - PAQ_XA^T = I - PAQ_XA^T\]

since \(Q_L = P^T\)

Also keeping in mind that \(A^TPA = Q_x^{-1}\), the matrix multiplication will give:

\[(PQ_v)(PQ_v) = (I - PAQ_XA^T)(I - PAQ_XA^T) = I - PAQ_XA^T - PAQ_XA^TPAQ_XA^T + PAQ_XA^TPAQ_XA^T = I - PAQ_XA^T - PAQ_XA^T + PAQ_XA^T = I - PAQ_XA^T = (PQ_v)\]

\[
MS \text{ rec'd 01/09/05} \\
Revised MS rec'd 02/06/04
\]

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